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Abstract. In this note we show that the Novikov-Veselov equation (NV-equation) at positive energy (an analog of KdV in 2+1 dimensions) has no exponentially localized solitons in the two- dimensional sense.

1.Introduction and Theorem 1. We consider the following 2+1 - dimensional analog of the KdV equaion (Novikov-Veselov equation):

$$\begin{aligned}&\partial_t v = 4Re\,(4\partial_z^3 v + \partial_z(vw) - E\partial_z w),\\ &\partial_{\bar z} w = -3\partial_z v, \quad v = \bar v, \quad E \in \mathbb{R},\\ &v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R},\end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \quad \partial_{\bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big). \tag{2}$$

We assume that

$$\begin{aligned}&v \ \text{ is sufficiently regular and has sufficient decay as } \ |x| \to \infty,\\ &w \ \text{ is decaying as } \ |x| \to \infty.\end{aligned} \tag{3}$$

Equation (1) is contained implicitly in the paper of S.V.Manakov [M] as an equation possessing the following representation

$$\frac{\partial(L-E)}{\partial t} = [L-E, A] + B(L-E) \tag{4}$$

(Manakov L-A-B- triple), where $L = -\Delta + v(x,t)$, $\Delta = 4\partial_z\partial_{\bar z}$, A and B are suitable differential operators of the third and zero order respectively, $[\cdot,\cdot]$ denotes the commutator. Equation (1) was written in an explicit form by S.P.Novikov and A.P.Veselov in [NV1], [NV2], where higher analogs of (1) were also constructed. Note also that the both Kadomtsev-Petviashvili (KP) equations can be obtained from (1) by considering an appropriate limit $E \to \pm\infty$, see [ZS], [G2].

For the case when

$$v(x_1, x_2, t), \ w(x_1, x_2, t) \ \text{ are independent of } \ x_2 \tag{5}$$

equation (1) is reduced to

$$\partial_t v = 2\partial_x^3 v - 12 v\partial_x v + 6E\partial_x v, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{6}$$

In terms of $u(x,t)$ such that

$$v(x,t) = u(-2t, x+6Et), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{7}$$

equation (6) takes the standard form of the KdV equation (see [NMPZ]):

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{8}$$

It is well-known (see [NMPZ]) that (8) has the soliton solutions

$$u(x,t) = u_{\kappa,\varphi}(x - 4\kappa^2 t) = -\frac{2\kappa^2}{ch^2(\kappa(x - 4\kappa^2 t - \varphi))}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \ \kappa \in]0,+\infty[, \ \varphi \in \mathbb{R}. \tag{9}$$

In addition, one can see that

$$\begin{aligned} &u_{\kappa,\varphi} \in C^\infty(\mathbb{R}), \\ &\partial_x^j u_{\kappa,\varphi}(x) = O(e^{-2\kappa|x|}) \ \text{ as } \ x \to \infty, \ j = 0,1,2,3,\ldots \end{aligned} \tag{10}$$

Properties (10) show, in particular, that the solitons of (9) are exponentially localized in x.

In the present note we obtain, in particular, the following result:

Theorem 1. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} &v(x,t) = V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ &V \in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(e^{-\alpha|x|}) \ \ for |x| \to \infty, \quad |j| \le 3 \ \ and\ some\ \alpha > 0, \end{aligned} \tag{11a}$$

(where $j = (j_1, j_2) \in (0 \cup \mathbb{N})^2$, $|j| = |j_1| + |j_2|$, $\partial_x^j = \partial^{j_1+j_2}/\partial x_1^{j_1}\partial x_2^{j_2}$),

$$w(\cdot,t) \in C(\mathbb{R}^2), \quad w(x,t) \to 0 \ \ as \ \ |x| \to \infty, \quad t \in \mathbb{R}. \tag{11b}$$

Then $V \equiv 0$, $v \equiv 0$, $w \equiv 0$.

Theorem 1 shows that equation (1) for $E > 0$ has no nonzero solitons (travel wave solutions) exponentially localized in x in the two-dimensional sense. For $E < 0$ this result will be given in [KN]. Note also that some other integrable systems in 2+1 dimensions admit exponentially decaying solitons in all directions on the plane, see [BLMP], [FS].

The proof of Theorem 1 is based on Proposition 1 and Proposition 2, see Section 4. In turn, Proposition 2 is based, in particular, on Lemma 1 and Lemma 2.

Lemma 1, Lemma 2 and Proposition 1 are recalled in Section 2. Proposition 2 is given in Section 3. It seems that the result of Proposition 2 (that sufficiently localized travel wave solutions for the NV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12)) was not yet formulated in the literature.

2. Lemma 1, Lemma 2 and Proposition 1. Consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E = E_{fix} > 0, \tag{12}$$

where

$$\begin{aligned}
&v(x) = \overline{v(x)}, \quad x \in \mathbb{R}^2, \\
&(1+|x|)^{2+\varepsilon} v(x) \in L^\infty(\mathbb{R}^2) \ \ \text{(as a function of } \ x \in \mathbb{R}^2) \ \ \text{for some} \ \ \varepsilon > 0.
\end{aligned} \tag{13}$$

It is known that for any $k \in \mathbb{R}^2$, such that $k^2 = E$, there exists an unique continuous solution $\psi^+(x,k)$ of equation (12) with the following asymptotics:

$$\psi^+(x,k) = e^{ikx} - i\pi\sqrt{2\pi}e^{-i\pi/4} f(k, |k|\frac{x}{|x|})\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o(\frac{1}{\sqrt{|x|}}) \ \ \text{as} \ \ |x| \to \infty. \tag{14}$$

This solution describes scattering of incident plane wave e^{ikx} on the potential v. The function f on

$$\mathcal{M}_E = \{k \in \mathbb{R}^2, \ l \in \mathbb{R}^2 : \ k^2 = l^2 = E\} \tag{15}$$

arising in (14) is the scattering amplitude for v in the framework of equation (12). Under assumptions (13), it is known, in particular, that

$$f \in C(\mathcal{M}_E). \tag{16}$$

Lemma 1. *Let v satisfy (13) and v_y, $y \in \mathbb{R}^2$, be defined by*

$$v_y(x) = v(x-y), \quad x \in \mathbb{R}^2. \tag{17}$$

Then the scattering amplitude f for v and the scattering amplitude f_y for v_y are related by the formula

$$f_y(k,l) = f(k,l)e^{iy(k-l)}, \quad (k,l) \in \mathcal{M}_E, \quad y = (y_1, y_2) \in \mathbb{R}^2. \tag{18}$$

Lemma 1 follows, for example, from the definition of the scattering amplitude by means of (14) and the fact that $\psi^+(x-y,k)$ solves (12) for v replaced by v_y, where $k^2 = E$.

Lemma 1 was given, for example, in [N3].

Lemma 2. *Let v, w satisfy (1), (3), where $E = E_{fix} > 0$. Then the scattering amplitude $f(\cdot,\cdot,t)$ for $v(\cdot,t)$ and the scattering amplitude $f(\cdot,\cdot,0)$ for $v(\cdot,0)$ are related by*

$$f(k,l,t) = f(k,l,0)\exp[2it(k_1^3 - 3k_1k_2^2 - l_1^3 + 3l_1l_2^2)], \quad (k,l) \in \mathcal{M}_E, \quad t \in \mathbb{R}. \tag{19}$$

Lemma 2 was given for the first time in [N1].

Note that in the framework of Lemma 2 properties (3) can be specified as follows:

$$\begin{aligned}
&v, w \in C(\mathbb{R}^2 \times \mathbb{R}) \ \ \text{and for each} \ \ t \in \mathbb{R} \ \ \text{the following properties are fulfiled :} \\
&v(\cdot,t) \in C^3(\mathbb{R}^2), \ \ \partial_x^j v(x,t) = O(|x|^{-2-\varepsilon}) \ \ \text{for} \ \ |x| \to \infty, \ \ |j| \le 3 \ \ \text{and some} \ \ \varepsilon > 0, \\
&w(x,t) \to 0 \ \ \text{for} \ \ |x| \to \infty.
\end{aligned} \tag{20}$$

Proosition 1. *Let*

$$v(x) = \overline{v(x)}, \quad e^{\alpha|x|}v(x) \in L^\infty(\mathbb{R}^2) \ \ \text{(as a function of } \ x) \ \ \text{for some} \ \ \alpha > 0 \tag{21}$$

and the scattering amplitude $f \equiv 0$ on $\mathcal{M}_E$ for this potential for some $E = E_{fix} > 0$. Then $v \equiv 0$ in $L^\infty(\mathbb{R}^2)$.

In the general case the result of Proposition 1 was given for the first time in [GN]. Under the additional assumption that v is sufficiently small (in comparison with E) the result of Proposition 1 was given for the first time in [N2]-[N4].

3. Transparency of solitons. In this section we show that sufficiently localized solitons (travel wave solutions) for the NV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12).

Proposition 2. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} &v(x,t) = V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ &V \in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(|x|^{-2-\varepsilon}) \ \ for |x| \to \infty, \quad |j| \le 3 \ \ and\ some\ \varepsilon > 0, \end{aligned} \tag{22a}$$

$$w(\cdot, t) \in C(\mathbb{R}^2), \quad w(x,t) \to 0 \ \ as \ \ |x| \to \infty, \quad t \in \mathbb{R}. \tag{22b}$$

Then

$$f \equiv 0 \ \ \text{on} \ \ \mathcal{M}_E, \tag{23}$$

where f is the scattering amplitude for $v(x) = V(x)$ in the framework of the Schrödinger equation (12).

The proof of Proposition 2 consists in the following.

We consider

$$T = \{\lambda \in \mathbb{C} : \quad |\lambda| = 1\}. \tag{24}$$

We use that

$$\mathcal{M}_E \approx T \times T, \quad E = E_{fix} > 0, \tag{25}$$

where diffeomorphism (25) is given by the formulas:

$$\lambda = \frac{k_1 + ik_2}{\sqrt{E}}, \quad \lambda' = \frac{l_1 + il_2}{\sqrt{E}}, \quad (k,l) \in \mathcal{M}_E, \tag{26}$$

$$\begin{aligned} k_1 &= \frac{\sqrt{E}}{2}\big(\lambda + \frac{1}{\lambda}\big), \quad k_2 = \frac{i\sqrt{E}}{2}\big(\frac{1}{\lambda} - \lambda\big), \\ l_1 &= \frac{\sqrt{E}}{2}\big(\lambda' + \frac{1}{\lambda'}\big), \quad l_2 = \frac{i\sqrt{E}}{2}\big(\frac{1}{\lambda'} - \lambda'\big), \quad (\lambda, \lambda') \in T \times T. \end{aligned} \tag{27}$$

We use that in the variables λ, λ' formulas (18), (19) take the form

$$f_y(\lambda, \lambda', E) = f(\lambda, \lambda', E) \exp\big[\frac{i}{2}\sqrt{E}\big(\lambda\bar{y} + \frac{y}{\lambda} - \lambda'\bar{y} - \frac{y}{\lambda'}\big)\big], \tag{28}$$

where $(\lambda, \lambda') \in T \times T$, y is considered as $y = y_1 + iy_2$,

$$f(\lambda, \lambda', E, t) = f(\lambda, \lambda', E, 0) \exp\big[iE^{3/2}t\big(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \big(\frac{1}{\lambda'}\big)^3\big)\big], \tag{29}$$

where $(\lambda, \lambda') \in T \times T$, $t \in \mathbb{R}$.

The assumptions of Proposition 2 and Lemmas 1 and 2 (with (18), (19) written as (28), (29)) imply that

$$\begin{gathered} f(\lambda, \lambda', E) \exp\big[\frac{i}{2}\sqrt{E}t\big(\lambda\bar{c} + \frac{c}{\lambda} - \lambda'\bar{c} - \frac{c}{\lambda'}\big)\big] = \\ f(\lambda, \lambda', E) \exp\big[iE^{3/2}t\big(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \big(\frac{1}{\lambda'}\big)^3\big)\big] \end{gathered} \tag{30}$$

for $(\lambda, \lambda') \in T \times T$, $t \in \mathbb{R}$, where f is the scattering amplitude for $v(x, 0) = V(x)$, c is considered as $c = c_1 + ic_2$.

Property (16), identity (30) and the fact that λ^3, λ^{-3}, λ, λ^{-1}, 1 are linear independent on each nonempty open subset of T imply (23).

4. Proof of Theorem 1 and final remark. Theorem 1 follows from Proposition 1 and Proposition 2.

Finally, note that the result of Theorem 1 does not hold, in general, without the assumption that $V(x) = O(e^{-\alpha|x|})$ as $|x| \to \infty$ for some $\alpha > 0$: "counter examples" to Theorem 1 with rational bounded V decaying at infinity as $O(|x|^{-2})$ are contained (in fact) in [G1], [G2]. As regards prototypical algebraically decaying solitons for KP1 equation, see [FA].

References

[BLMP] M.Boiti, J.J.-P.Leon, L.Martina, F.Pempinelli, Scattering of localized solitons in the plane, Physics Letters A **132**(8,9), 432-439 (1988)

[FA] A.S.Fokas, M.J.Ablowitz, On the inverse scattering of the time-dependent Schrödinger equation and the associated Kadomtsev- Petviashvili (I) equation, Studies in Appl. Math. **69**, 211-228 (1983)

[FS] A.S.Fokas, P.M.Santini, Coherent structures in multidimensions, Phys.Rev.Lett. **63**, 1329-1333 (1989)

[G1] P.G.Grinevich, Rational solitons of the Veselov-Novikov equation - Two-dimensional potentials that are reflectionless for fixed energy, Teoret. i Mat. Fiz. **69**(2), 307-310 (1986) (in Russian); English translation: Theoret. and Math.Phys. **69**, 1170-1172 (1986).

[G2] P.G.Grinevich, The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy, Uspekhi Mat.Nauk **55**(6), 3-70 (2000) (in Russian); English translation: Russian Math. Surveys **55**(6), 1015-1083 (2000)

[GN] P.G.Grinevich, R.G.Novikov, Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials. Cmmun.Math.Phys. **174**, 400-446 (1995).

[KN] A.V.Kazeykina, R.G.Novikov, Absence of exponentially localized solitons for the Novikov-Veselov equation at negative energy, (in preparation)

[M] S.V.Manakov, The inverse scattering method and two-dimensional evolution equations, Uspekhi Mat.Nauk **31**(5), 245-246 (1976) (in Russian)

[N1] R.G.Novikov, Construction of a two-dimensional Schrödinger operator with a given scattering amplitude at fixed energy, Teoret. i Mat Fiz **66**(2), 234-240 (1986) (in Russian); English translation: Theoret. and Math.Phys. **66**, 154-158 (1986).

[N2] R.G.Novikov, Reconstruction of a two-dimensional Schrödinger operator from the scattering amplitude at fixed energy, Funkt.Anal. i Pril. **20**(3), 90-91 (1986) (in Russian); English translation: Funct.Anal. and Appl. **20**, 246-248 (1986).

[N3] R.G.Novikov, Inverse scattering problem for the two-dimensional Schrödinger equation at fixed energy and nonlinear equations, PhD Thesis, Moscow State University 1989 (in Russian)

[N4] R.G.Novikov, The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator, J.Funct.Anal. **103**, 409-463 (1992).

[NMPZ] S.Novikov, S.V.Manakov, L.P.Pitaevskii, V.Z.Zakharov, Theory of solitons: the inverse scattering method, Springer, 1984.

[NV1] S.P.Novikov, A.P.Veselov, Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolution equations. Dokl.Akad.Nauk. SSSR **279**, 20-24 (1984) (in Russian); English translation: Sov.Math.Dokl. **30**, 588-591 (1984).

[NV2] S.P.Novikov, A.P.Veselov, Finite-zone, two-dimensional Schrödinger operators. Potential operators. Dokl.Akad.Nauk. SSSR **279**, 784-788 (1984) (in Russian); English translation: Sov.Math.Dokl. **30**, 705-708 (1984).

[ZS] V.E.Zakharov, E.I.Shulman, Integrability of nonlinear systems and perturbation theory / / What is integrability? Berlin: Springer-Verlag, 185-250 (1991)